Finite Volume Approximation of MHD Equations with Euler Potential

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Abstract

The possibility to produce energy by fusion reactions is being studied in experimental devices called tokamaks where charged particles are confined in a toroidal vacuum chamber thanks to a very large magnetic field. The ITER device currently build in Cadarache (France) will be the largest tokamak ever realized for these experiments. The large scale dynamics of charged particles in a tokamak as ITER can be described by the Magneto-HydroDynamics equations (MHD). This system of equations contains as an involution the divergence-free constraint of the magnetic field, $\nabla \cdot \mathbf{B} = 0$ that has to be maintained by the numerical approximation.

The respect of the divergence-free constraint can be achieved in two different ways. The first class consists in adding to the MHD system penalization terms ensuring that the magnetic field will be solenoidal. The second class consists in formulating the MHD system in term of the vectorial potentiel $\mathbf{A}$ that satisfies $\nabla \times \mathbf{A} = \mathbf{B}$, fulfilling thus automatically the divergence-free constraint.

The proposed method is a formulation of the MHD system in term of the mixture of the two former classes. The resulting system is divergence-free constraint preserving and can be approximated by standard Finite Volume methods. Various numerical tests on well-known standard problems in MHD show that this approach is an interesting alternative and opens possibility to use a conservative formulation based on $\mathbf{B}$ of the MHD system.

MHD equations with Euler potential

Definitions

- ψ: Euler potential,
- ρ: density,
- $\mathbf{u}$: velocity,
- $\mathbf{B} = B_z \mathbf{e}_z + \mathbf{e}_z \times \nabla\psi$: magnetic field,
- p: pressure,
- $p^* = p + \frac{1}{2}\mathbf{B}^2$: total pressure,
- $E = \frac{p}{\gamma-1} + \frac{1}{2}\rho\mathbf{u}^2 + \frac{1}{2}\mathbf{B}^2$: total energy.

Conservative equations with $\partial_z \cdot = 0$

- MHD system:

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho\mathbf{u}) & = 0, \\ \partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u} - \mathbf{B}\otimes\mathbf{B}) + \nabla p^* & = 0, \\ \partial_t E + \nabla\cdot[(E+p^*)\mathbf{u} - (\mathbf{u}.\mathbf{B})\mathbf{B})] & = 0, \\ \partial_t\mathbf{B} + \nabla\cdot(\mathbf{u}\otimes\mathbf{B} - \mathbf{B}\otimes\mathbf{u}) & = 0. \end{cases}$$

- Euler potential equation:

$$\partial_t(\rho\psi) + \nabla\cdot(\rho\psi\mathbf{u}) = 0.$$

- Free-divergence constraint:

$$\nabla\cdot\mathbf{B} = 0.$$

Previous works

- Vector potential $\mathbf{A}$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

Insure $\nabla\cdot\mathbf{B} = 0$.
Drawback: One order higher in spatial derivatives.

- Divergence cleaning methods
One example: Generalized Lagrange Multiplier (GLM) [1, 3]

$$\begin{aligned} \partial_t\mathbf{B} + \nabla\cdot(\mathbf{u}\otimes\mathbf{B} - \mathbf{B}\otimes\mathbf{u}) + \nabla\Psi &= 0, \\ \mathcal{D}(\Psi) + \nabla\cdot\mathbf{B} &= 0, \\ \mathcal{D}(\Psi) &= \frac{1}{c_h^2}\partial_t\Psi + \frac{1}{c_p^2}\Psi. \end{aligned}$$

Easily incorporated with a Riemann type scheme.
Drawback : $\nabla\cdot\mathbf{B}$ appears in several equations and they are not approximated with the same discretization. Those methods usually insure $\nabla\cdot\mathbf{B} = 0$ for one discrete approximation.

Proposed method and numerical scheme

Proposed method: Mixture of the two previous methods
Work with a redundant system: MHD equations + Potential equation.

$$\partial_t U + \partial_x F(U) + \partial_y G(U) = 0.$$

Numerical Scheme: based on relaxation scheme in two steps.

- Transport step: finite volume method

$$U_{i,j}^{n+1/2} = U_{i,j}^n - \frac{\Delta t}{\Delta x}(F_{i+1/2,j}^n - F_{i-1/2,j}^n) - \frac{\Delta t}{\Delta y}(G_{i,j+1/2}^n - G_{i,j-1/2}^n),$$

$F_{i+1/2,j}^n$, $G_{i,j+1/2}^n$: numerical fluxes computated with HLLD scheme [2].

- Projection step :

$$\begin{cases} \mathbf{B}_{i,j}^{n+1} = B_{z,i,j}^{n+1/2}\mathbf{e}_z + \mathbf{e_z}\times(\nabla\psi)_{i,j}^{n+1/2}, \\ E_{i,j}^{n+1} = \frac{p_{i,j}^{n+1/2}}{\gamma-1} + \frac{1}{2}\rho_{i,j}^{n+1/2}(\mathbf{u}_{i,j}^{n+1/2})^2 + \frac{1}{2}(\mathbf{B}_{i,j}^{n+1})^2. \end{cases}$$

$(\nabla\psi)_{i,j}^{n+1/2}$: computed with centered finite differences.

Orszag-Tang problem

- Initial data: $\gamma = 5/3$

ρ	$u(x,y)$	$v(x,y)$	$p(x,y)$	$B_x(x,y)$	$B_y(x,y)$	$\psi(x,y)$
γ^2	$-\sin(2\pi y)$	$\sin(2\pi x)$	γ	$-\sin(2\pi y)$	$\sin(4\pi x)$	$-\frac{1}{2\pi}\cos(2\pi y) - \frac{1}{4\pi}\cos(4\pi x)$

- Pressure field

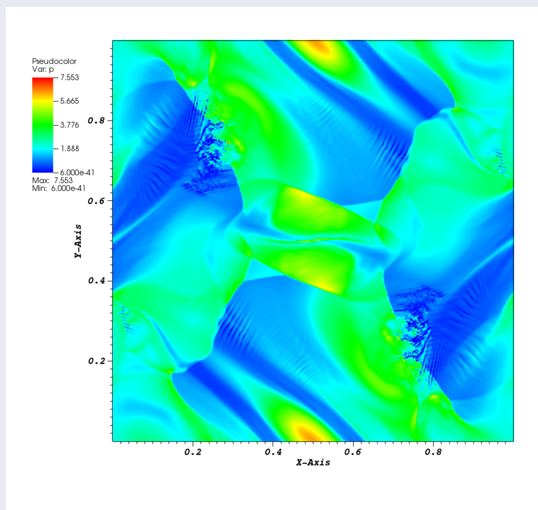


Figure : Without projection.

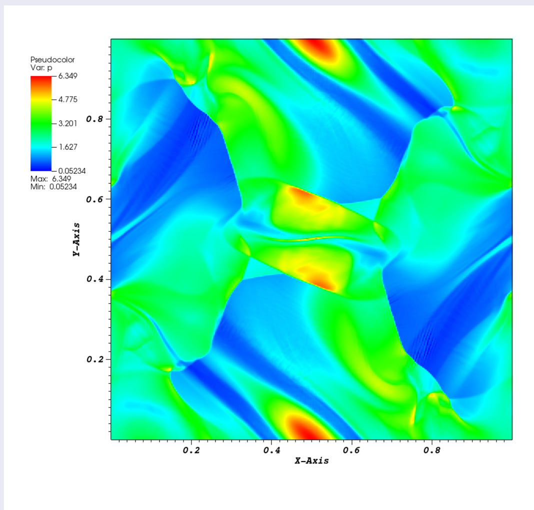
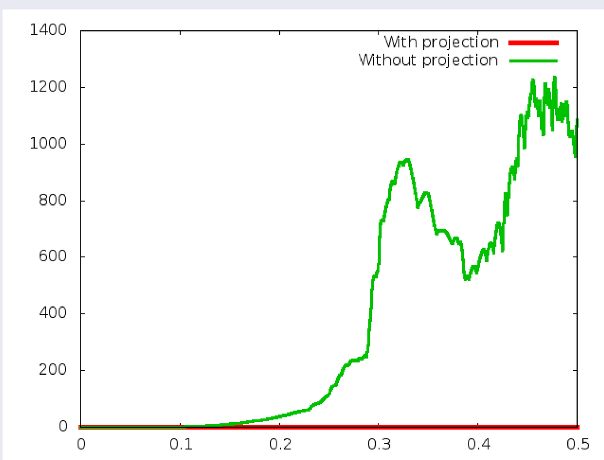


Figure : With projection.

- Divergence of the magnetic field



Kelvin-Helmoltz instabilities

- Initial data:

ρ	p	$u(x,y)$	$v(x,y)$	$B_x(x,y)$	$B_y(x,y)$	$B_z(x,y)$	$\psi(x,y)$
1	$\frac{1}{\gamma}$	$\frac{1}{2}\tanh(\frac{y}{y_0})$	0	$0.1\cos(\frac{\pi}{3})\sqrt{\rho}$	0	$0.1\sin(\frac{\pi}{3})\sqrt{\rho}$	$-0.1\cos(\frac{\pi}{3})\sqrt{\rho}y$

- Single mode pertubation

$$v(x,y) = 0.01\sin(2\pi x)\exp(-\frac{y^2}{\sigma^2}),\ \sigma = 0.01.$$

- $B_{pol}/B_{tor} = \sqrt{B_x^2 + B_y^2}/B_z$ at $t = 5.0$

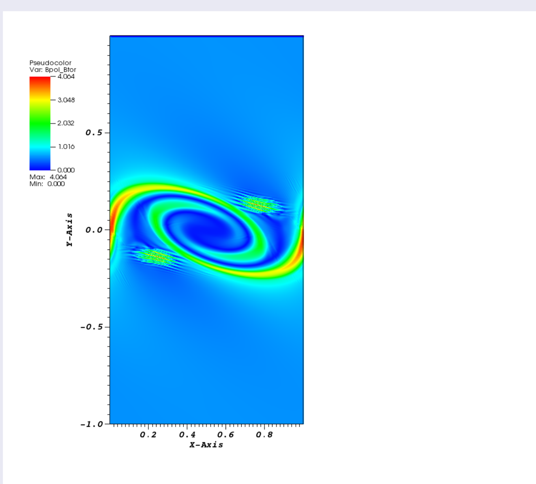


Figure : Without projection.

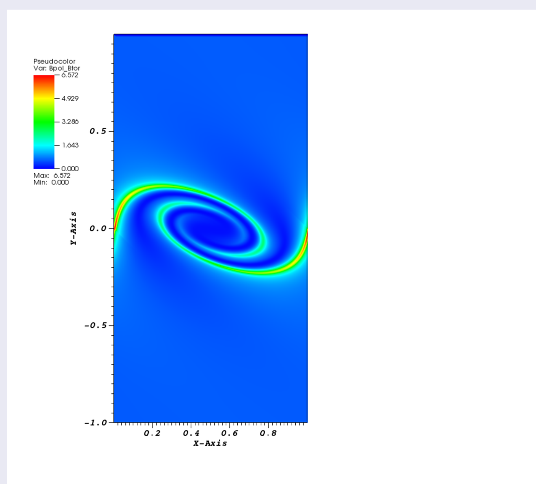


Figure : With projection.

Kelvin-Helmoltz instabilities

- Divergence of the magnetic field:

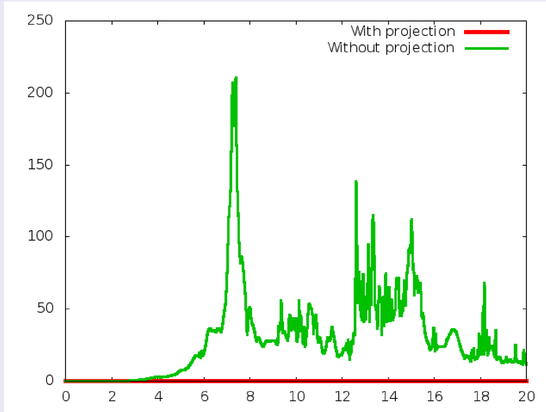


Figure : $||\nabla\cdot\mathbf{B}||_\infty$.

Screw pinch equilibrium

- Initial data:

$$\begin{cases} R_0 = 10, & B_r = 0, \\ \rho = 1, & B_\theta(r) = \frac{r}{R_0(3r^2+1)}, \\ \mathbf{u} = 0, & B_z = 1, \\ p(r) = \frac{1}{6R_0^2(3r^2+1)^2}, & \psi(r) = \frac{1}{6R_0}\ln(3r^2+1). \end{cases}$$

- Cylindrical coordinates:

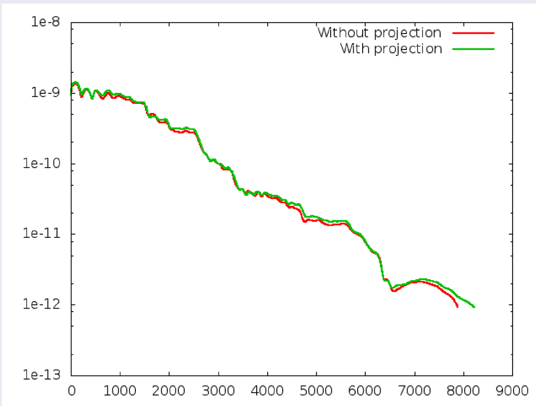


Figure : Residual on r-momentum equation.

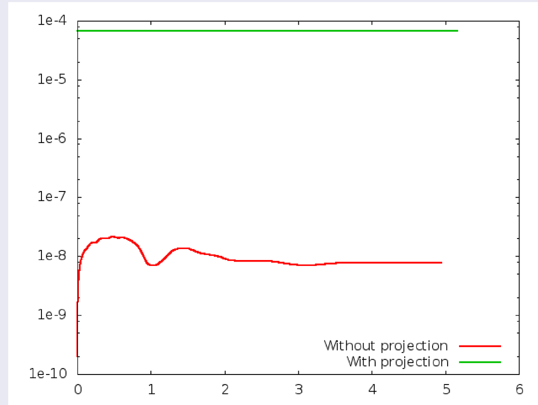


Figure : Relative error of B_θ.

- Cartesian coordinates:

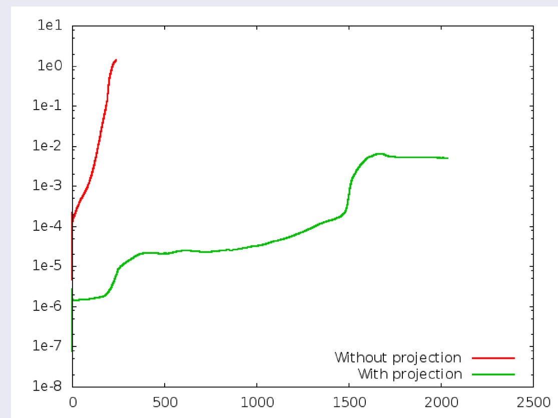


Figure : Relative error of p in function of Alfvén time.

Conclusions and perspectives

Conclusions:

- Shock capturing tests: Scheme with projection gives satisfactory results.
- Plasma fusion test: Work in progress on well-balanced scheme.

Perspectives:

- Perform more tests for plasma fusion.
- Test on resistive MHD problems.

References

[1] A. Dedner, F. Kemm, D. Kröner, C-D. Munz, T. Schnitzer, and M. Wesenberg *Hyperbolic Divergence Cleaning for the MHD Equations*, JCP 175, 645-673, 2002.

[2] T. Miyoshi, and K. Kusano *A multi-state HLL approximate Riemann solver for ideal magneto-hydrodynamics*, JCP, 208(1) 314:344, 2005.

[3] C-D. Munz, P. Omnes, R.Schneider, E. Sonnendrücker, and U. Voß *Divergence Correction Techniques for Maxwell Solvers Based on a Hyperbolic Model*, JCP, 161:484-511, 2000.